MATH 427 | 527

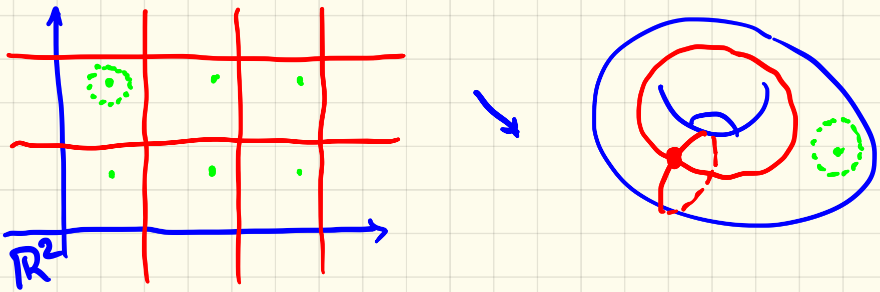
NOW WE'RE GOING TO FOCUS ON MANIFOLDS.

THESE ARE HAUSDORFF SPACES M FOR WHICH:

FOR EACH $x \in M$ THERE'S AN OPEN $x \in U_x \subset M$
FOR WHICH $U_x \cong \mathbb{R}^n$

THESE ARE "n-MANIFOLDS" OR "MANIFOLDS OF DIMENSION n". FOR NOW, WE'LL ASSUME COMPACTNESS; A COMPACT MANIFOLD IS "CLOSED".

<u>EXP</u> 1 SURFACES ARE 2-MANIFOLDS.



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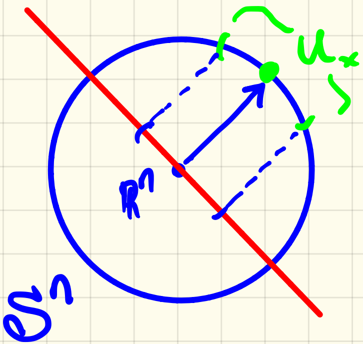


S^n IS AN n-MANIFOLD.

THE INTEGER $n = \dim(M)$ IS INTRINSIC :

$$H_i(M, M \setminus \{x\}) \underset{\substack{\uparrow \\ \text{EXCISION} \\ \text{WITH } U_x \cong \mathbb{R}^n}}{\cong} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \underset{\substack{\uparrow \\ \text{L.E.S of } (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ {}^*\mathbb{R}^n \text{ IS CONTRACTIBLE}^*}}{\cong} \tilde{H}_{i-1}(\mathbb{R}^n \setminus \{0\}) \underset{\substack{\uparrow \\ S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}}}{\cong} \tilde{H}_{i-1}(S^{n-1})$$

NOW $\tilde{H}_{i-1}(S^{n-1}) \cong \mathbb{Z}$ IFF $i-1 = n-1$

OR

$$\boxed{H_n(M, M \setminus \{x\}) \neq 0 \text{ ONLY WHEN } n = \dim(M)}.$$

IF M IS A (CLOSED) ORIENTABLE MANIFOLD, THEN POINCARÉ DUALITY ASSERTS:

$$H_k(M) \cong H^{n-k}(M)$$

RMK THIS WILL HOLD FOR ALL $k \in \mathbb{Z}$. FOR INSTANCE:

$$H_{*>\dim(M)}(M) = 0 = H^{*<0}(M)$$

RMK WE WON'T SHOW THIS, BUT $H_k(M)$ IS FINITELY GENERATED. AS A RESULT OF OUR "UNIVERSAL COEFICIENT THEOREM" I.E

$$f: H^k(M) \twoheadrightarrow \mathrm{Hom}(H_k(M), \mathbb{Z})$$

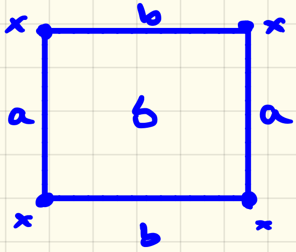
WE HAVE:

$$H_k(M) \cong H_{n-k}(M) \quad \text{MODULO TORSION}$$

$$H_k(M) \cong H_{n-k-1}(M) \quad \text{ON TORSION SUBGROUPS.}$$

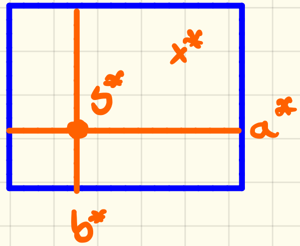
INTUITION. CONSIDER $S^1 \times S^1$

X:



BUILD A DUAL CELL STRUCTURE

X^*:



LET C BE CHAINS: $\langle \sigma \rangle \longrightarrow \langle a, b \rangle \longrightarrow \langle x \rangle$

C^* BE DUALS: $\langle x^* \rangle \longrightarrow \langle a^*, b^* \rangle \longrightarrow \langle \sigma^* \rangle$

THIS IDENTIFICATION OF COMPLEXES IS CANONICAL UP TO SIGN. [WANT CANONICAL? TAKE $\mathbb{Z}/2\mathbb{Z}$ INSTEAD OF $\mathbb{Z}$].

C BE CHAINS: $\langle \overset{2}{\sigma} \rangle \longrightarrow \langle \overset{1}{a,b} \rangle \longrightarrow \langle \overset{0}{x} \rangle$

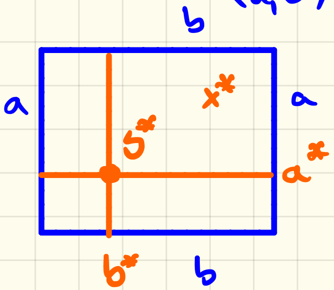
C^* BE DUALS: $\langle x^* \rangle \longrightarrow \langle a^*, b^* \rangle \longrightarrow \langle \sigma^* \rangle$

$d: C_i \longrightarrow C_{i-1}$ ASSIGNS A SUM OF $(i-1)$-CELL THAT ARE FACES OF AN i-CELL.

$\delta: C^*_{2-i} \longrightarrow C_{2-i+1}$ ASSIGNS A "SUM OF CELLS OF WHICH IT IS A FACE" TO A CELL.

EXP $d(\sigma) = a+b-a-b$ AND $\delta(x^*) = a^* + b^* - a^* - b^*$.

IN FACT THE $\eta: \underset{\langle a,b\rangle}{C_1} \longrightarrow \mathbb{Z}$ ARE CHARACTERIZED



$\eta^a \quad a^*(a) = 1 \qquad a^*(b) = 0$

$\eta^b \quad b^*(a) = 0 \qquad b^*(b) = 1$

(NOW EXTEND LINEARLY).

$\langle \eta^a, \eta^b \rangle \cong H^1(S^1 \times S^1)$.

RMK THE UNDERLYING SIGN ISSUE IN THE ABOVE CONSTRUCTION IS ULTIMATELY TAKEN CARE OF BY THE ORIENTATION.

WHAT IS AN ORIENTATION?

DEF AN ORIENTATION OF $\mathbb{R}^n$ AT A POINT $x \in \mathbb{R}^n$ IS A CHOICE OF GENERATOR OF

$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}$$

CONSIDER A SPHERE S^{n-1} CENTRED AT $x \in \mathbb{R}^n$ SO THAT

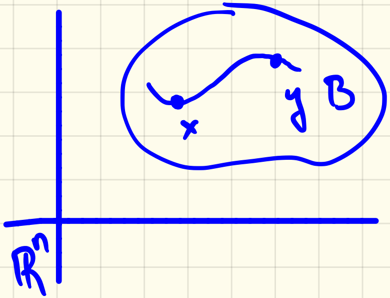
$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H_{n-1}(\mathbb{R}^n \setminus \{x\}) \cong H_{n-1}(S^{n-1})$$

WE'VE SEEN: ROTATIONS HAVE DEGREE 1 — FIXES GENERATOR

REFLECTIONS HAVE DEGREE -1 — FLIPS GENERATOR

RMK FOR $n=3$ THIS IS EXACTLY WHAT THE BEHAVIOR OF THE "RIGHT HAND RULE" e.g. ORIENTATIONS IN $\mathbb{R}^3$.

THIS NOTION HAS THE "RIGHT" FLEXIBILITY:



SUPPOSE $B \in \mathbb{R}^n$ (CONNECTED) BALL WITH $x, y \in B$

$$\begin{tikzcd} & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) & \\ H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \arrow[ur, "x \in B", "\cong"'] \arrow[rr, dashed, leftrightarrow, "\cong"'] & & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}) \arrow[ul, "y \in B"', "\cong"] \end{tikzcd}$$

<u>DEF</u> A "LOCAL ORIENTATION" AT $x \in M$ IS A CHOICE OF GENERATOR

$$\langle \mu_x \rangle \cong H_n(M, M \setminus \{x\}).$$

HATCHER'S NOTATION:

$$H_n(X \mid A) := H_n(X, X \setminus A)$$

THIS IS THE "LOCAL HOMOLOGY OF X AT $A \subset X$".

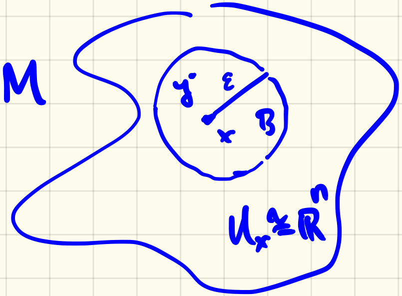
IN PARTICULAR: LOCAL ORIENTATION $\mu_x \in H_n(M \mid x)$.

DEF An "orientation" of a manifold M is function

$$x \longmapsto \mu_x$$

assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M|x)$ satisfying "local consistency":

in M



There a ball B of finite radius with $x \in B$ so that

$$\mu_x = \mu_B = \mu_y$$

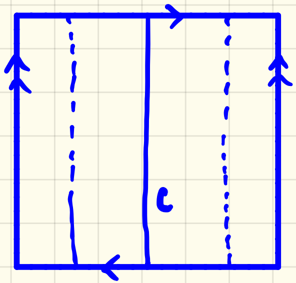
for all $y \in B$.

In particular, μ_B determines all the μ_y under

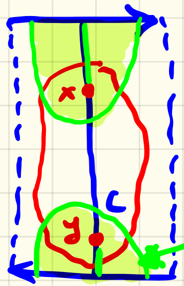
$$H_n(M|B) \xrightarrow{\cong} H_n(M|y).$$

If an orientation exists, M is "orientable".

EXP



K THE KLEIN BOTTLE



LOCAL CONSISTENCY WILL BREAK HERE

NOTICE THAT THIS IS SOMEWHAT SIMILAR TO/ REMINISCENT OF OUR COHOMOLOGICAL ARGUMENT ABOUT "IMPOSIBLE FIGURES". CAN YOU BUILD AN ASSOCIATED COCYCLE (TRY MOVING ALONG c)?

AS YOU KNOW: K IS 2-FOLD COVERED BY THE TORUS T. INDEED

$$\boxed{\tilde{T} \cong \tilde{K} \cong \mathbb{R}^2} \, .$$

$$\begin{tikzcd}
 & \mathbb{R}^2 \arrow[dl] \arrow[dr] & \\
T \arrow[rr, dashed] & & K
\end{tikzcd}$$

IN GENERAL, EVERY M HAS AN ORIENTABLE DOUBLE-COVER $\overline{M} = \{ \mu_x \mid \mu_x$ LOCAL ORIENTATION OF M AT $x \}$

AS A SET

BY CONSTRUCTION $\overline{M} \twoheadrightarrow M$ IS A 2-TO-1 SURJECTION.

$\mu_x \longmapsto x$

PUT A TOPOLOGY ON $\overline{M}$: $B \subset \mathbb{R}^n \subset M$ AND A GENERATOR $\mu_B \in H_n(M|B)$

LET $U(\mu_B) = \{ \mu_x \in \overline{M} \mid x \in B$ AND $H_n(M|B) \xrightarrow{\cong} H_n(M|x)$, $\mu_B \longmapsto \mu_x \}$

THIS GIVES A BASIS OF OPEN SETS. THE SURJECT. $p: \overline{M} \to M$ IS A COVERING SPACE MAP:

$$p^{-1}(U_x) = U(\mu_x) \amalg U(-\mu_x)$$

EACH $\mu_x \in \tilde{M}$ HAS A CANONICAL LOCAL ORIENT. GIVEN BY $\tilde{\mu}_x \in H_n(\tilde{M} \mid \mu_x)$ VIA

$$H_n(\tilde{M} \mid \mu_x) \cong H_n(U(\mu_B) \mid \mu_x) \cong H_n(B \mid x)$$

$$\tilde{\mu}_x \longleftrightarrow \mu_x$$

EXC CHECK LOCAL CONSISTENCY.

FOR M CONNECTED THEN:

HATCHER 3.25

M ORIENTABLE $\Longleftrightarrow$ $\tilde{M}$ HAS TWO COMPONENTS.

(IN PARTICULAR, SIMPLY-CONNECTED MANIFOLDS ARE ORIENTABLE.)

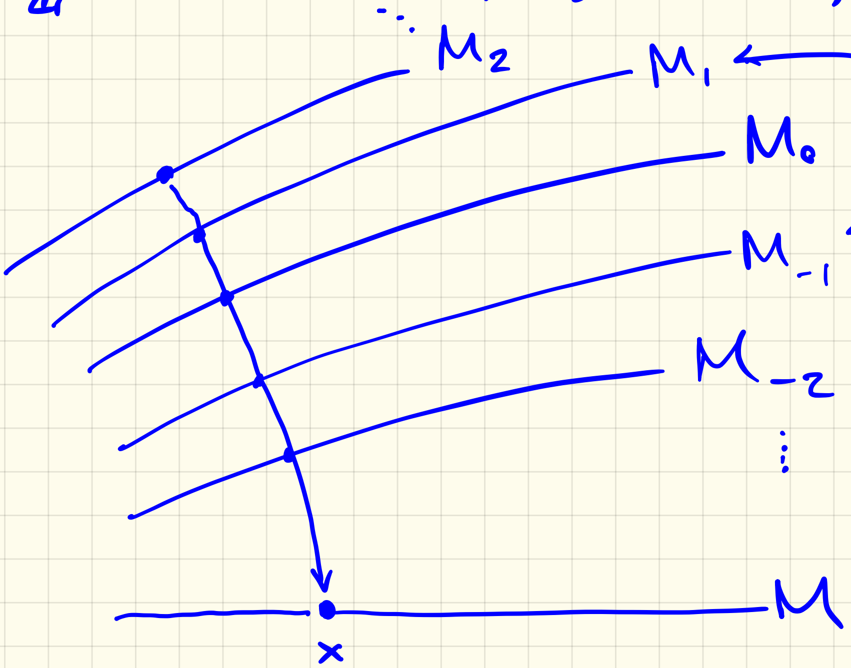
THIS FITS INTO A MORE GENERAL SET-UP:

$\tilde{M} \to M$ EMBEDS IN A LARGER COVER

$$M_{\mathbb{Z}_1} \longrightarrow M$$

WHERE $M_{\mathbb{Z}} = \{\alpha_x \in H_n(M|x) \mid x \in M\}$.

$M_{\mathbb{Z}}$ IS INFINITELY SHEETED, LOCALLY:



$\overline{M} = M_1 \cup M_{-1}$ LOOSELY

$\boxed{M_0 \cong M}$

THE MAP $M \longrightarrow M_{\mathbb{Z}}$ TAKING M TO M_0 IS CALLED THE 'ZERO SECTION'.

A "SECTION" IS A MAP $M \longrightarrow M_{\mathbb{Z}}$

$x \longmapsto \alpha_x$

SO AN ORIENTATION OF M IS THE SAME THING AS A SECTION $x \longmapsto \mu_x$ WHERE EACH $\langle \mu_x \rangle \cong H_n(M | x)$

HATCHER 3.26

THM IF M IS ORIENTABLE THEN $H_n(M) \xrightarrow{\cong} H_n(M|x)$

DEF AN ELEMENT OF $H_n(M)$ WHOSE IMAGE IN $H_n(M|x)$ IS A GENERATOR FOR ALL x IS CALLED A "FUNDAMENTAL CLASS".

THM THE TORSION SUBGROUP OF $H_{n-1}(M)$

HATCHER 3.28

$$\cong \begin{cases} 0 & M \text{ ORIENTABLE.} \\ \mathbb{Z}/2\mathbb{Z} & M \text{ NOT ORIENTABLE.} \end{cases}$$

LET M BE AN n-MANIFOLD, WHICH WE ASSUME IS CLOSED.

LAST TIME: DEFINED $H_n(M|x) := H_n(M; M \setminus \{x\}) \cong \mathbb{Z}$

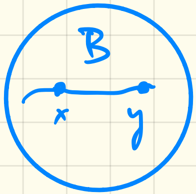
A "LOCAL ORIENTATION" OF M IS A FUNCTION

$$x \longmapsto \mu_x$$

WHERE $\langle \mu_x \rangle \cong H_n(M|x)$ FOR ALL $x \in M$ SATISFYING LOCAL CONSISTENCY: IF $x, y \in B$, WHERE B IS AN n-BALL

THEN

$$\begin{array}{ccccc} \mu_x & = & \mu_B & = & \mu_y \\ \cap & & \cap & & \cap \\ H_n(M|x) & & H_n(M|B) & & H_n(M|y) \\ & & \wr & & \\ & & H_n(M; M \setminus B) & & \\ & & \wr & & \\ & & \mathbb{Z} & & \end{array}$$



DEF M IS "ORIENTABLE" IF SUCH A FUNCTION EXISTS.

THM IF M IS ORIENTABLE THEN $H_n(M) \xrightarrow{\cong} H_n(M|x)$ $\forall x \in M$.

DEF AN ELEMENT OF $H_n(M)$ WHOSE IMAGE IN $H_n(M|x)$ IS A GENERATOR FOR ALL $x \in M$ IS A "FUNDAMENTAL CLASS" FOR M.

PRP IF A FUNDAMENTAL CLASS EXISTS, M IS ORIENTABLE.

PROOF GIVEN $\mu \in H_n(M)$ SUCH THAT $\mu \mapsto \mu_x$ AND

$$\langle \mu_x \rangle \cong H_n(M|x)$$

NOTE THAT

$$\begin{tikzcd} H_n(M) \arrow[rr, "\cong"] \arrow[dr, "\star"'] & \mu \mapsto \mu_x \arrow[d, phantom, "\circlearrowleft"] & H_n(M|x) \\ & H_n(M|B) \arrow[ur, "\cong"'] & \end{tikzcd}$$

WHERE B IS A BALL AND $x \in B$.

BY CONSTRUCTION, ISOMORPHISM FACTORS THROUGH $H_n(M|B)$. THIS DIAGRAM VERIFIES LOCAL CONSISTENCY $\boxtimes$.

AS A RESULT "FUNDAMENTAL CLASS" $\equiv$ "ORIENTATION CLASS".

WE'RE TRYING TO UNDERSTAND | ESTABLISH $H_k(M) \cong H^{n-k}(M)$ FOR <u>ORIENTABLE</u> M.

THE ISOMORPHISM IS GENERATED BY THE "CAP PRODUCT"

WE'LL WORK WITH SINGULAR HOMOLOGY, SO THAT:

$$C_n(X) \text{ GENERATED BY } \sigma : \Delta^n \longrightarrow X$$

$$\varphi \in \mathrm{Hom}(C_n(X), \mathbb{Z}) = C^n(X) \text{ ASSIGNS } \varphi(\sigma : \Delta^n \to X) \in \mathbb{Z}.$$

RECALL: $\delta\varphi(\sigma) = \sum_i (-1)^i \varphi(\sigma|[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n])$.

SO IF $\varphi \in C^n(X)$ AND $\psi \in C^m(X)$ THEN

$$(\underset{\substack{\uparrow \\ n+m}}{\varphi \smile \psi)(\sigma)} = \underbrace{\varphi(\sigma|\underbrace{[v_0, \ldots, v_n]}_{n}) \psi(\sigma|\underbrace{[v_n \ldots v_{n+m}]}_{m})}_{\mathbb{Z}}.$$

SIMILARLY: $\sigma \in C_n(X)$, $\varphi \in C^m(X)$ $n \geq m$.

$$\sigma \frown \varphi := \underbrace{\varphi\left(\sigma|[v_0, v_1 \dots v_m]\right)}_{\text{NEW COEFFICIENT}} \cdot \underbrace{\sigma|[v_m, \dots, v_n]}_{\Delta^{n-m} \to X}$$

So $\sigma \frown \varphi \in C_{n-m}(X)$. THIS INDUCES A PRODUCT ON HOMOLOGY; DEFINE

$$\partial(\sigma \frown \varphi) = (-1)^m \left(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi\right).$$

LET'S THINK THROUGH THIS MORE FORMALLY:

$$\begin{aligned}
C^m(X) \otimes C_n(X) &\xrightarrow{id \otimes \Delta_*} C^m(X) \otimes C_n(X \times X) \\
\varphi \otimes \sigma \quad &\xrightarrow{id \otimes \psi} C^m(X) \otimes \left(\bigoplus C_i(X) \otimes C_{n-i}(X)\right) \\
&\xrightarrow{\ \ast\ } \underbrace{C^m(X) \otimes C_m(X)}_{\mathbb{Z}} \otimes C_{n-m}(X) \\
&\longrightarrow C_{n-m}(X) \\
&\qquad\ \ \varphi \frown \sigma
\end{aligned}$$

RMK THE 'EVALUATION' $\ast$ IS SOMETHING WE'VE SEEN:

$$f : H^n(X) \twoheadrightarrow \mathrm{Hom}(H_n(X), \mathbb{Z})$$

THE INDUCED (COMPOSITE) MAP IS THE "CAP":

$$\frown : H^m(X) \otimes H_n(X) \longrightarrow H_{n-m}(X) \qquad n \geq m.$$

<u>EXC</u> CHECK THAT IF φ, σ GENERATE, $\varphi \otimes \sigma$ IS AN ELEMENTARY TENSOR, AND $\varphi \frown \sigma$ IS A CYCLE.

NOTE THAT IF $f : X \to Y$ WE GET A FUNNY INDUCED EXPRESSION:

$$f_*(\sigma) \frown \varphi = f_*\left(\sigma \frown f^*(\varphi)\right)$$

THAT IS:

$$\begin{tikzcd}
\sigma \quad H_n(X) \otimes H^m(X) \arrow[r, "\frown"] \arrow[d, "f_*"'] & H_{n-m}(X) \quad \sigma \frown f^*(\varphi) \arrow[d, "f_*"] \\
H_n(Y) \otimes H^m(Y) \quad \varphi \arrow[u, "f^*"', shift right=8] \arrow[r, "\frown"] & H_{n-m}(Y) \quad f_*(\sigma) \frown \varphi
\end{tikzcd}$$

NOW FOR M AN ORIENTABLE MANIFOLD:

$$PD : H^k(M) \longrightarrow H_{n-k}(M)$$
$$\varphi \longmapsto [M] \cap \varphi$$

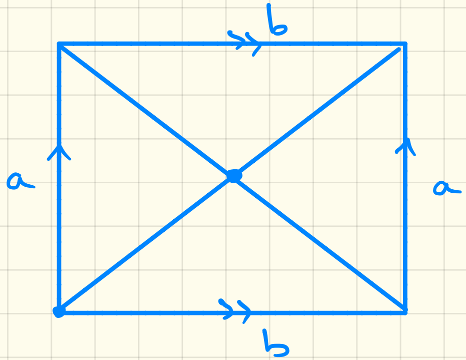
WHERE $[M]$ IS THE FUNDAMENTAL CLASS FOR M i.e

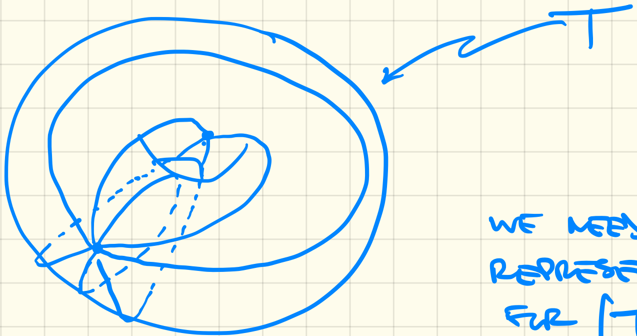
$$\langle [M] \rangle \cong H_n(M).$$

<u>THM</u> [POINCARÉ DUALITY] $PD = [M] \cap -$ IS AN ISOMORPHISM.

<u>EXAMPLE: THE 2-TORUS.</u>

SEE HATCHER 3.31



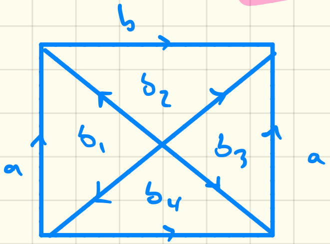


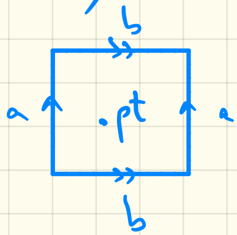
WE NEED A REPRESENTATIVE FOR $[T]$.

(GENERATING $H_2(T) \cong \mathbb{Z}$).

ON CELLULAR HOMOLOGY THIS SHOULD BE / HAS TO BE

$$\langle [e^2] \rangle \cong C_2(S^1 \times S^1) \cong \mathbb{Z} \quad (\cong H_2(S^1 \times S^1))$$

$T =$ 

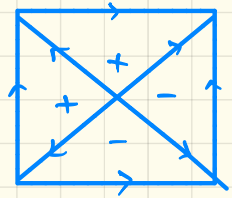


$$\left(\text{i.e. } e^2 = PD(pt) \right)$$

CONSIDER $b = b_1 + b_2 - b_3 - b_4 \in C_2(T) \cong \mathbb{Z}^4 \ni (1,1,-1,-1)$

NOTE THAT $d(b_1 + b_2 - b_3 - b_4) = a + b - a - b = 0$

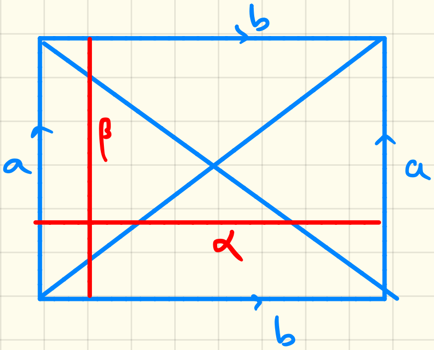
SO THAT b IS A CYCLE, HENCE $\langle b \rangle = [T]$



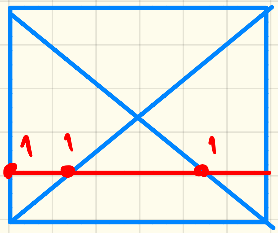
NOW WE ALSO HAVE $C_1(T) \cong \mathbb{Z}^6$

$$H_1(T) \cong \langle a, b \rangle$$

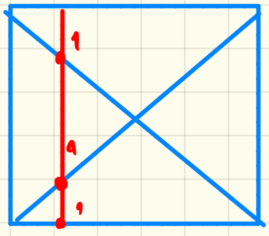
AND WE HAVE $\quad H^1(T) \cong \mathrm{Hom}(H_1(T), \mathbb{Z})$



$$\alpha \longmapsto (a \mapsto 1,\ b \mapsto 0)$$
$$\beta \longmapsto (a \mapsto 0,\ b \mapsto 1)$$



$\varphi : C_1 \to \mathbb{Z}$



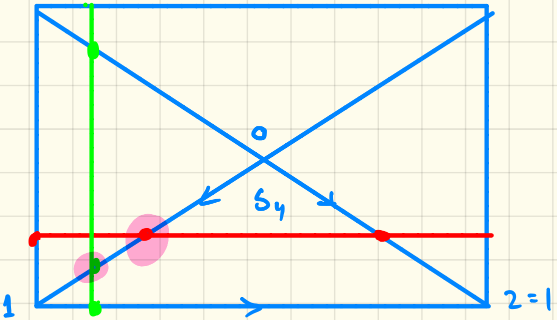
$\varphi : C_1 \to \mathbb{Z}$

NOW CONSIDER THE MAP $[T] \frown - : H^1(T) \longrightarrow H_1(T)$

DEFINITION: $[T] \frown \varphi = \sigma \frown \varphi = \varphi(\sigma|[v_0, v_1]) \cdot [v_1, v_2]$

WHAT DOES THIS MEAN? $[v_0, v_1, v_2]$ (ORDERED) 2-SIMPLEX AND WE WANT ONE(S) SUCH THAT φ IS NON-ZERO ON $[v_0, v_1]$.

$$b \cap \varphi = b_1 \cap \varphi + b_2 \cap \varphi - b_3 \cap \varphi - \underbrace{b_4 \cap \varphi}$$



$$- \underbrace{\varphi(b_4 | [v_0\, v_1])}_{1} \underbrace{[v_1, v_2]}_{b}$$

$$\boxed{b \cap \varphi = -b}$$

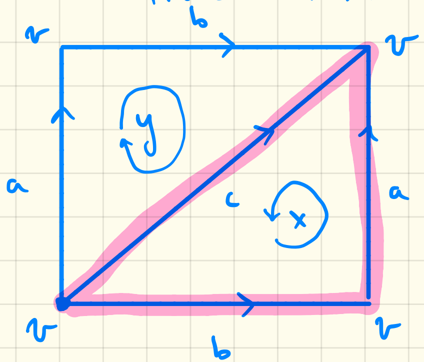
$b = [v_1, v_2]\ (= [v_0, v_0])$

$$b \cap \psi \ (= b_1 \cap \psi) = \underbrace{\psi(b_1 | [v_0\, v_1])}_{1} \underbrace{[v_1\, v_0]}_{a} = a$$

$\underline{\underline{\text{So}}}$ $-b$ POINCARÉ DUAL TO α

a —— " —— β

CHECK THIS AGAIN ...



$$0 \longrightarrow \langle x, y\rangle \longrightarrow \langle a,b,c\rangle \xrightarrow{0} \langle v\rangle \longrightarrow 0$$

$$a,b,c \longmapsto v-v$$

$$x \longmapsto b+a-c$$

$$y \longmapsto -c+b+a$$

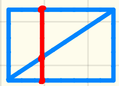
$\mathbb{Z} \cong \langle x-y\rangle$ $\qquad$ $\langle a,b,c\rangle \big/ c=a+b$ $\qquad$ $\mathbb{Z}$

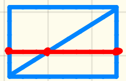
SO $[T] = x - y$, LET $\eta : C_1(T) \to \mathbb{Z}$.

NOW: $[T] \cap \eta = x \cap \eta - y \cap \eta$

$$= \eta(b)\,a - \eta(a)\,b$$

$$= \begin{cases} +a & \text{FOR} \\ -b & \text{FOR} \end{cases}$$

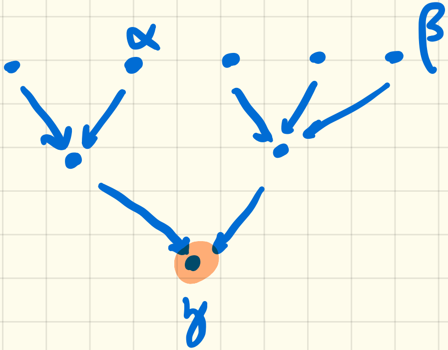




LET I BE A "DIRECTED SET"

THIS MEANS: I IS A SET WITH A PARTIAL ORDER $\leq$ SATISFYING, FOR EACH PAIR $\alpha, \beta \in I$ $\exists \gamma \in I$ FOR WHICH $\alpha \leq \gamma$ AND $\beta \leq \gamma$.

EXP ROOTED TREES: TREES T WITH DISTINGUISHED "ROOT" VERTEX CALLED γ:



$$\alpha \longrightarrow \beta \iff \alpha \leq \beta$$

SUPPOSE THAT $G_\alpha \in \mathrm{abGrp}$ FOR $\alpha \in I$ AND FOR $\alpha, \beta \in I$ MORPHISM

$$f_{\alpha\beta} : G_\alpha \longrightarrow G_\beta \quad \text{WHENEVER } \alpha \leq \beta$$

SUCH THAT (1) $f_{\alpha\alpha} = \mathbb{1}_{G_\alpha}$

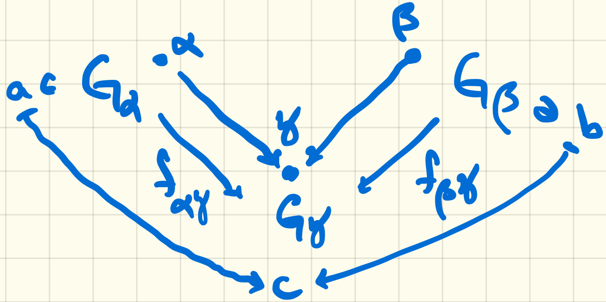
(2) $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$ FOR $\alpha \leq \beta \leq \gamma$

THIS DATA $\{G_\alpha\}$ IS A "DIRECTED SYSTEM" IN abGrp.

<u>DEF</u> $\varinjlim G_\alpha = \left. \bigoplus_\alpha G_\alpha \middle/ \sim \right.$ WHERE $\sim$ IS GIVEN BY $a = f_{\alpha\beta}(a)$

THIS SHOULD OUTPUT A GROUP; HERE'S AN EQUIV. BUT MORE MANAGEABLE VIEW:

LET $\underset{\substack{\cap \\ G_\alpha}}{a} \sim \underset{\substack{\cap \\ G_\beta}}{b}$ IF $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$ FOR SOME γ



NOTE THAT $[a]+[b]=[a'+b']$ WHERE $a'\in[a]$, $b'\in[b]$

SUCH $a',b'\in G_\gamma$.

ADDITION IN G_γ.

THIS GIVES $\cancel{\coprod_\alpha G_\alpha /_\sim} = G$

A GROUP STRUCTURE SO $G\in \mathrm{abGrp}$.

<u>EXC</u> $G=\varinjlim G_\alpha$.

<u>EXP</u> $G_\alpha=\mathbb{Z}$ FOR $\alpha\in\mathbb{N}$ (NOTE TOTAL ORDER $\Rightarrow$ DIRECTED SET)

AND $f_{\alpha\beta}=\times 1$

THEN $[x]\in G$ $\quad x=(x_1,x_2,x_3,x_4\cdots)$

$f_{12}(x_1)=x_2=x_1$

$\left([x]=[y] \iff x_i=y_i \ \forall\, i\gg 0 \ \ldots \text{ IN FACT } \forall\, i\right)$

CHECK: $G=\varinjlim G_\alpha=\mathbb{Z}$.

FACT: IF $J \subset I$ SUCH THAT $\forall \alpha \in I \; \exists \beta \in J$ FOR WHICH

$$\alpha \leq \beta$$

THEN $$\varinjlim_{I} G_\alpha = \varinjlim_{J} G_\alpha$$

SO IF I HAS A MAXIMAL ELEMENT, γ SAY, THEN

$$J = \{\gamma\}$$

AND
$$\varinjlim_{I} G_\alpha = \varinjlim_{\{\gamma\}} G_\gamma = G_\gamma .$$

NOW SUPPOSE X IS A SPACE WITH $X_\alpha \subset X$

$$X_\alpha \subset X_\beta \iff \alpha \leq \beta$$

SO
$$H_i(X_\alpha) \xrightarrow{\iota_{\alpha\beta}} H_i(X_\beta) \quad \left(\iota_{\alpha\beta} = i_* \right)$$

THIS GIVES A DIRECTED SYSTEM IN abGrp.

PRP IF EVERY COMPACT SET IN X IS CONTAINED IN SOME X_α THEN

$$\varinjlim H_i(X_\alpha) \cong H_i(X)$$

NOTE:

$$\begin{tikzcd}
\cdots H_i(X_\alpha) \arrow[rr, "i_{\alpha\beta}"] \arrow[dr, "i_*"'] & & H_i(X_\beta) \cdots \arrow[dl, "i_*"] \\
& H_i(X) &
\end{tikzcd}$$

WE WANT TO USE THESE LIMITS TO DEFINE "COHOMOLOGY WITH COMPACT SUPPORT"

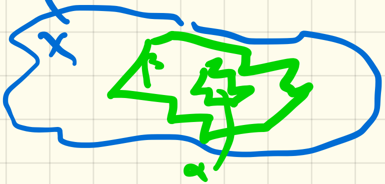
LET X BE A SPACE AND LET $K \subset X$ BE COMPACT.

$$K_\alpha \subset K_\beta \iff \alpha \leq \beta$$

CONSIDER $H^i(X, X \setminus K)$ IF $K_\alpha \subset K_\beta$ THEN

$$(X, X\setminus K_\beta) \hookrightarrow (X, X \setminus K_\alpha)$$

$$H^i(X, X\setminus K_\beta) \longleftarrow H^i(X, X\setminus K_\alpha)$$



DEF $H^i_c(X) = \varinjlim H^i(X, X \setminus K_\alpha)$

RMK IF X IS COMPACT THEN $H^i_c(X) = H^i(X)$

MAXIMAL COMPACT

FOR INSTANCE: A CLOSED MANIFOLD

EXP CONSIDER $H^*_c(\mathbb{R}^n) = \varinjlim H^*(\mathbb{R}^n, \mathbb{R}^n \setminus K)$

EVERY COMPACT K IS CONTAINED IN SOME CLOSED BALL B_r CENTRED AT $\vec{0}$ OF RADIUS $r \geq 0$.

SO: THIS PICKS OUT $J \subset I$ $(J = \mathbb{R}_{\geq 0})$

MAY AS WELL TAKE $r \in \mathbb{N}$.

ONLY INT. CASE!

$$\underbrace{H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_r)}_{H^n(\mathbb{R}^n | B_r)} \xrightarrow{\cong} \underbrace{H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_{r+1})}_{H^n(\mathbb{R} | B_{r+1})}$$

SO $H^n_c(\mathbb{R}^n) \cong \mathbb{Z}, \left(\neq H^n_c(pt) = H^n(pt) \right)$
$n > 0$

NOT AN INVARIANT OF HOMOTOPY TYPE.

(CHECK $H^{i \neq n}_c(\mathbb{R}^n) = 0$).

THIS MACHINERY LETS US DEFINE DUALITY FOR NON-COMPACT ORIENTABLE MANIFOLDS:

$$\begin{array}{ccccc} H_n(M|K_\beta) & \times & H^k(M|K_\beta) & \xrightarrow{\frown} & \\ {\scriptstyle i_*}\downarrow & & \uparrow{\scriptstyle i_*} & & H_{n-k}(M) \\ H_n(M|K_\alpha) & \times & H^k(M|K_\alpha) & \xrightarrow{\frown} & \end{array}$$

* NOTICE THAT $\mu_\alpha = i_*(\mu_\beta)$ FOR LOCAL ORIENTATIONS *

THM PD: $H^k_c(M) \longrightarrow H_{n-k}(M)$ IS AN ISOMORPHISM
HATCHER 3.35 FOR ALL k WHEN M IS ORIENTABLE MANIFOLD.

NOTE: WHEN M IS COMPACT, THIS IS DESIRED STATEMENT.

ONE OF THE KEY STEPS IN THE PROOF IS THE "LOCAL" CASE:

WHEN $\mathbb{R}^n = M$ REGARD $\mathbb{R}^n$ AS $\mathrm{int}(\Delta^n)$ SO THAT

$$[\Delta^n] \cap - : H^k(\Delta^n, \partial\Delta^n) \xrightarrow{PD} H_{n-k}(\Delta^n)$$

IN THE CASE $k = n$: $H^n(\Delta^n, \partial\Delta^n) \cong \mathrm{Hom}(H_n(\Delta^n, \partial\Delta^n), \mathbb{Z})$

$$\cong \langle \varphi \mid \varphi(\Delta^n) = 1 \rangle$$

UNPACK DEFINITION OF CAP PRODUCT:

$\Delta^n \cap \varphi$ IS THE LAST VERTEX OF Δ^n WHICH REPRESENTS A GENERATOR OF $H_0(\Delta^n)$

(A): IF $M = A \cup B$ IS THE UNION OF OPEN SETS AND PD_A, PD_B, $PD_{A \cap B}$ ARE ISOMORPHISMS THEN

$$PD_M$$

IS AN ISOMORPHISM TOO.

(B) IF M IS A UNION OF $U_1 \subset U_2 \subset \cdots$ AND EACH

$$PD_{U_i} : H^k_c(U_i) \xrightarrow{\cong} H_{n-k}(U_i)$$

THEN SO IS PD_M.

(HERE USE $H_{n-k}(M) \cong \varinjlim H_{n-k}(U_i)$).

[READING: FULL PROOF IN HATCHER].

<u>CRL</u> IF M IS CLOSED ORIENTABLE n-MANIFOLD AND n IS ODD THEN $\chi(M) = 0$.

HATCHER 3.37

<u>PROOF</u> $\operatorname{rank} H_i(M) = \operatorname{rank} H^{n-i}(M) = \operatorname{rank} H_{n-i}(M)$

JUST COMPUTE OVER $\mathbb{Q}$.

SO $\chi(M) = \sum_i (-1)^i \operatorname{rank} H_i(M)$

IF n <u>ODD</u>

$$\overset{?}{=} \sum_i \pm \left(\operatorname{rank} H_i(M) - \operatorname{rank} H_{n-i}(M)\right)$$

$= 0$. ∎

OTHER FORMS.

THM IF K IS COMPACT AND LOCALLY CONTRACTIBLE IN M

HATCHER 3.44 THEN

$$H_i(M, M \setminus K) \cong H^{n-i}(K)$$

CRL [ALEXANDER DUALITY] IF K COMPACT, NON-EMPTY, PROPER,

HATCHER 3.45 LOCALLY CONTRACTIBLE IN S^n THEN

$$H_{i+1}(S^n, S^n \setminus K) \cong \tilde{H}_i(S^n \setminus K) \cong \tilde{H}^{n-i-1}(K)$$

FOR MOST i

TWO EXAMPLES RELEVANT TO OUR COURSE:

JUST AN S^1

① SUPPOSE $K \hookrightarrow S^3$ IS A KNOT THEN

$$H_1(S^3 \setminus K) \cong H^1(K) \cong \mathbb{Z} \cong H_1(K)$$

② SUPPOSE $F \hookrightarrow S^3$ IS A (SEIFERT) SURFACE SO THAT $\partial F = K$.

$$H_1(S^3 \setminus F) \cong H^1(F) \cong H_1(F) \cong \mathbb{Z}^{2g(F)}$$